Exercises for seminar week 41

Supplementary exercise 4 (*read the introduction below*)
Rice, chapter 4: No. 69, 75, 76, 77, 79, 94
(For 94 read section 4.6 in Rice and (A4-5) in appendix 1 in "Lecture notes to Rice chapter 5" on the net.)

Hint for ex 4.79: Remember the sum of a geometric series:

$1+a+a^2+a^3+\cdots=\sum_{i=0}^{\infty}a^i=\frac{1}{1-a}$ for all numbers, *a*, such that $|a|<1$.

A common factor in such a series can be taken outside the sum as for finite sums:

$\sum_{i=0}^{\infty}ca^i=c\sum_{i=0}^{\infty}a^i$

Introduction to supplementary exercise 4

Assume $X\sim\Gamma(\alpha,\lambda)$. In the exercise (and elsewhere in the course) we need a formula for $\mathrm{E}(X^r)$ where *r* is any real number such that *r* > 0: The density for *X* is

$$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}\text{ for }x>0,\text{ and }f(x)=0\text{ for }x\le 0.$$

The task becomes easy when we remember that $\int_{-\infty}^{\infty}f(x)dx=\int_0^{\infty}f(x)dx=1$, for all $\alpha,\lambda>0$. (Note that the integral from $-\infty$ to 0 is equal to 0 here). We find

$$\mathrm{E}(X^r)=\int_{-\infty}^{\infty}x^r f(x)dx=\int_0^{\infty}x^r f(x)dx=\frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^{\infty}x^{r+\alpha-1}e^{-\lambda x}dx$$

By multiplying and dividing by the same constant, we can write the integral as an integral of a *pdf* (i.e. the *pdf* of $\Gamma(r+\alpha,\lambda)$), which has value 1:

$$\mathrm{E}(X^r)=\frac{\lambda^\alpha}{\Gamma(\alpha)}\cdot\frac{\Gamma(r+\alpha)}{\lambda^{r+\alpha}}\cdot\int_0^{\infty}\frac{\lambda^{r+\alpha}}{\Gamma(r+\alpha)}x^{r+\alpha-1}e^{-\lambda x}dx=\frac{\lambda^\alpha}{\Gamma(\alpha)}\cdot\frac{\Gamma(r+\alpha)}{\lambda^r\lambda^\alpha}\cdot 1$$

Hence

$$\mathrm{E}(X^r)=\frac{\Gamma(r+\alpha)}{\lambda^r\Gamma(\alpha)}\quad\text{for any real } r>0.$$

(For example, $\mathrm{E}\left(\sqrt{X}\right)=\frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha)}\cdot\frac{1}{\sqrt{\lambda}}$)